

The University of Chicago

CONJUGATE SYSTEMS
CHARACTERIZED BY SPECIAL
PROPERTIES OF THEIR RAY
CONGRUENCES

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1932

By

EMMA JULIA OLSON

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INTRODUCTION

We consider conjugate systems which are characterized by special properties of their ray congruences. We assume that the ray congruence is given, impose conditions upon it, and study the conjugate net which determines the congruence and the surface sustaining the conjugate net.

We make use of Wilczynski's classic invariant projective theory. In so far as possible, the notation is the same as that used by Wilczynski in the papers referred to as references 1 and 2 in the bibliography.

The two sections of the first chapter consist of fundamental projective theory which is used in the succeeding chapters. Section 1 contains the analytic basis of the work. Section 2 contains known properties of an integral surface, of its Laplace transformed surfaces, and of curves on these surfaces. The contents of section 2 are derived chiefly from Wilczynski's Brussels paper.

The two sections of the second chapter include the derivations of systems of differential equations of the ray congruence.

In the sections of the two remaining chapters are obtained necessary and sufficient conditions that a focal sheet of the ray congruence degenerate into a point, a curve, and a straight line. These conditions are used in finding the nature of the conjugate net determining the ray congruence and of the surface sustaining the net when the following conditions are imposed upon the ray congruence; namely, that the ray congruence be a bundle of lines, a ruled plane, a congruence with degenerate focal sheets, a proper linear congruence, and a singular linear congruence. The first four cases are discussed in the third chapter. The fourth chapter is devoted to conjugate systems characterized by linear ray congruences.

Some of the results obtained are the following. If a ray congruence is a bundle of lines, the conjugate net determining the congruence lies on a cone with the vertex of the cone at the center of the bundle. If the rays are the lines of a ruled plane, the conjugate net which determines the ray congruence consists of cone curves on a non-developable surface and the ray curves of the net are indeterminate. If the focal sheets of the ray congruence degenerate into curves which are distinct, the surface sustaining the given conjugate net is in general non-ruled, and the ray curves on the surface are cone curves. If, in this case, the ray curves coincide with either the asymptotic net or the

associate conjugate net, the given surface is a quadric. If the focal sheets of the ray congruence degenerate into curves which are coincident, the surface sustaining the given conjugate net is in general non-ruled, but it may be a developable surface with the coincident focal curves as its edge of regression and the ray curves as its generators, or a ruled surface with the ray curves coinciding with one family of asymptotics. If the foci of the rays of a proper linear congruence separate the ray points harmonically and if the ray curves form a conjugate net, the given surface and all of its non-degenerate Laplace transforms are quadrics with isothermally conjugate parametric nets. A conjugate net on a non-developable surface with non-degenerate first and minus first Laplace transforms is isothermally conjugate if it determines a ray congruence consisting of lines intersecting a single straight line.

CHAPTER I

FUNDAMENTAL PROJECTIVE THEORY

1. Analytic basis. In this section we give the analytic basis of our work. The general projective theory of rectilinear congruences may be studied [1]* by a method in which use is made of a completely integrable system of partial differential equations of the form

$$\begin{aligned} y_v &= mz, & z_u &= ny, \\ (D) \quad y_{uu} &= ay + bz + cy_u + dz_v, \\ z_{vv} &= a'y + b'z + c'y_u + d'z_v, \end{aligned}$$

where the subscripts indicate partial differentiations and the coefficients are assumed to be analytic functions of the independent arguments u and v . If the coefficients satisfy the conditions

*The number 1 in the symbol [1] means reference 1 in the bibliography at the end of the text. The symbol may include a paragraph reference, as [1, § 10], or a page reference, as [1, p. 28].

$$c = f_u, \quad d' = f_v, \quad b = -d_v - df_v, \quad a' = -c'_u - c'f_u,$$

$$W = mn - c'd = f_{uv},$$

$$(I) \quad \begin{aligned} m_{uu} + d_{vv} + df_{vv} + d_v f_v - f_u m_u &= ma + db', \\ n_{vv} + c'_{uu} + c'f_{uu} + c'_u f_u - f_v n_v &= nb' + c'a, \\ 2m_u n + mn_u &= a_v + f_u mn + a'd, \\ m_v n + 2mn_v &= b'_u + f_v mn + bc', \end{aligned}$$

where f is an analytic function of u and v , system (D) is completely integrable and has exactly four pairs of linearly independent solutions $(y^{(i)}, z^{(i)})$, $(i = 1, 2, 3, 4)$ such

that the general solution is of the form

$$y = \sum_{i=1}^4 c^{(i)} y^{(i)}, \quad z = \sum_{i=1}^4 c^{(i)} z^{(i)},$$

where $c^{(1)}, c^{(2)}, c^{(3)}, c^{(4)}$ are arbitrary constants.

The $y^{(i)}$ and $z^{(i)}$ may be interpreted* as the homogeneous coordinates of two points P_y and P_z of ordinary space. As u and v vary, these two points describe two surfaces S_y and S_z either or both of which may be degenerate. The line $P_y P_z$ then describes a congruence for which S_y and S_z are the focal sheets. The developables of the congruence are obtained by equating u or v to a constant. The developables determine a conjugate system of curves on each non-degenerate focal sheet. We shall assume from now on that the surface S_y is non-degenerate. Therefore, the points $P_y, P_{y_u},$ and P_{y_v} are distinct

* We shall usually write the coordinates of any point, say P_y , merely as y .

and non-collinear and the condition

$$m \neq 0$$

is satisfied.

The most general transformation which leaves the form of system (D) unchanged is

$$(T) \quad y = \lambda(u)\bar{y}, \quad z = \mu(v)\bar{z}, \quad \bar{u} = \alpha(u), \quad \bar{v} = \beta(v),$$

where $\lambda, \mu, \alpha, \beta$ are arbitrary functions of their arguments. Besides m, n, c', d some of the invariants of this group are I, J, l, l' defined by

$$I = a + (\log m)_u [c - (\log m)_u] - (\log m)_{uu},$$

$$J = b' + (\log n)_v [d' - (\log n)_v] - (\log n)_{vv},$$

$$l = b + d(\log n)_v, \quad l' = a' + c'(\log m)_u.$$

Four fundamental covariants are y, z , and ρ, σ defined by

$$\rho = y_u - (\log m)_u y, \quad \sigma = z_v - (\log n)_v z.$$

The axis of the point P_y on the surface S_y has been defined [2, p. 314] for a given conjugate net on the surface as the line of intersection of the osculating planes of the two curves of the net through P_y . The totality of these lines is called the axis congruence of the given conjugate net on S_y . The curves on the surface S_y which correspond to the developables of the axis congruence are called the axis curves. These curves are represented [2, p. 316] by the equation

$$d, du^2 - m I du dv - c'd m dv^2 = 0,$$

where the invariant d , is given by

$$d, = d[c'd - (\log d)_{uv}].$$

The ray of the point P_y on the surface S_y has been

defined [2, p. 318] for a given conjugate net on S_y , as the line $P_\rho P_z$, which joins corresponding points of the first and minus first Laplace transformed nets, and which corresponds dualistically to the axis of P_y . The totality of rays is called the ray congruence. The curves on the surface S_y which correspond to the developables of the ray congruence are called the ray curves.

The ray curves on the surface S_y defined with reference to the given conjugate net on S_y are represented [2, p. 318] by the equation

$$dm \, du^2 - m \, I \, du \, dv - m \, dv^2 = 0,$$

where the invariant m , is given by

$$m, = mk = m[mn - (\log m)_{uv}].$$

The foci of the ray are obtained from the covariant

$$(1) \quad dm, z^2 + m \, I \, z \rho - mn \rho^2.$$

If $n \neq 0$, the coordinates of the foci of the ray may be written in the form

$$\rho_1 = \rho + \lambda_1 z, \quad \rho_2 = \rho + \lambda_2 z,$$

where

$$2n \lambda_1 = -I + (I^2 + 4dkn)^{1/2}, \quad 2n \lambda_2 = -I - (I^2 + 4dkn)^{1/2}.$$

If $n = 0$, $I \neq 0$, the covariant (1) becomes $dm, z^2 + m \, I \, z \rho$.

Then the coordinates of one focus are $\rho + (dk/I)z$ and of the other focus are z . If $n = I = 0$, $dk \neq 0$, the coordinates of both foci are z . If $n = I = 0$, $dk = 0$, the coordinates of the foci are indeterminate.

In obtaining the integrability conditions (I), the points y, z, y_u, z_v are assumed to be distinct and not coplanar. Since the points z and z_v are distinct, the surface S_z neither degenerates into a u -curve nor into a fixed point. Since the points y, z, y_u, z_v are not coplanar, the surfaces S_y and S_z cannot be planes. A further condition imposed upon the surface $S_y(S_z)$ is that it cannot be a developable surface with the $v(u)$ -curves as generators. The reason for this restriction is found in section 2.

The integrability conditions (I) may be written in terms of invariants of the system (D) as follows

$$\begin{aligned}
 (I') \quad c &= f_u, \quad d' = f_v, \quad b = -d_v - df_v, \quad a' = -c'_u - c'f_u, \\
 W &= mn - c'd = f_{uv}, \\
 -l_v &= mI + dJ + l(\log n)_v, \\
 -l'_u &= nJ + c'I + l'(\log m)_u, \\
 k_u &= I_v + dl' + k[c - 2(\log m)_u], \\
 (n_{-1}/n)_v &= J_u + c'l + (n_{-1}/n)[d' - 2(\log n)_v],
 \end{aligned}$$

where n is different from zero and

$$n_{-1} = n[mn - (\log n)_{uv}].$$

2. The Laplace transformations. We consider in this section an integral surface S_y of system (D). Some of the known properties of the surface S_y , of the Laplace transformed surfaces defined by the parametric conjugate net on S_y , briefly called Laplace transforms of S_y , and of curves on these surfaces will be given here for reference. We shall indicate the method of deriving a few of the properties stated.

The Laplace sequence

$$\dots, z_{-1}, z, y, y, \dots,$$

we shall write as follows

$$\dots, \sigma, z, y, \rho, \dots.$$

The differential equations of the surface S_y are obtained from the system (D) by elimination of the variable z and its partial derivatives. These equations are

$$(Y) \quad \begin{aligned} y_{uu} &= (d/m)y_{vv} + cy_u + [(b/m) - (\log m)_v d/m]y_v + ay, \\ y_{uv} &= (\log m)_u y_v + mny. \end{aligned}$$

If $d = 0$, the surface S_y is a developable [1, p. 28]. Furthermore, if $d = 0$, then $b = 0$ from the third integrability condition of system (D). The first equation of system (Y) becomes a second order differential equation with respect to u only. Therefore, if $d = 0$, the u -curves on the developable surface S_y are the generators of this developable.

The differential equation of the asymptotic net on the surface S_y is [1, p. 55]

$$d^2 du + m^2 dv = 0.$$

If $d = 0$, the asymptotic net degenerates into the one-parameter family of straight lines which have just been shown to be the u -curves.

Let us assume from now on that d is different from zero, unless otherwise specified, and let us refer the surface S_y to its asymptotic net by applying the transformation

$$\bar{u} = \zeta(u, v), \quad \bar{v} = \eta(u, v) \quad (\zeta_u \eta_v - \zeta_v \eta_u \neq 0),$$

with

$$\zeta_u = (-d/m)^{1/2} \zeta_v, \quad \eta_u = (-d/m)^{1/2} \eta_v.$$

After performing the necessary calculations, the details of which need not be given here, we obtain two differential equations of the form

$$(1) \quad \begin{aligned} y_{\bar{u}\bar{u}} + 2\bar{a}y_{\bar{u}} + 2\bar{b}y_{\bar{v}} + \bar{c}y &= 0, \\ y_{\bar{v}\bar{v}} + 2\bar{a}'y_{\bar{u}} + 2\bar{b}'y_{\bar{v}} + \bar{c}'y &= 0, \end{aligned}$$

where

$$(2) \quad 2\zeta_v^2 \bar{b} = (-m/d)^{1/2} [B^{(y)} - (-d/m)^{1/2} C^{*(y)}] \eta_v,$$

$$(3) \quad 2\eta_v^2 \bar{a}' = -(-m/d)^{1/2} [B^{(y)} + (-d/m)^{1/2} C^{*(y)}] \zeta_v,$$

$$(4) \quad 8B^{(y)} = 2f_u - (\log m^3 d)_u, \quad 8C^{*(y)} = 2f_v + (\log m d^3)_v.$$

The surface S_y is, in the most general case, a non-ruled surface. If a surface is ruled, one of the two families of curves of the asymptotic net consists of straight lines. From equations (1), (2) and (3) we obtain either

$$(5) \quad B^{(y)} - (-d/m)^{1/2} C^{*(y)} = 0$$

or

$$(6) \quad B^{(y)} + (-d/m)^{1/2} C^{*(y)} = 0$$

as a necessary and sufficient condition for the surface S_y to be ruled.

If the surface S_y is a quadric, all of the asymptotics are straight lines. In this case the equations (5) and (6) hold. Therefore, the surface S_y is a quadric if, and only if,

$$B^{(y)} = C^{*(y)} = 0,$$

that is if, and only if,

$$(7) \quad f_u = (\log m^{3/2} d^{1/2})_u, \quad f_v = -(\log m^{1/2} d^{3/2})_v.$$

The equation (7) may be integrated to obtain

$$(8) \quad f = \log v^{-4} m^{3/2} d^{1/2} = -\log U^4 m^{1/2} d^{3/2},$$

where U is a function of u alone, and V is a function of v alone. From the last of equations (8) we have

$$(9) \quad \log m^{1/2} d^{1/2} = \log U^{-1} V.$$

By substitutions from the equation (9) into the two expressions for f in equations (8), we obtain

$$m = UV^3 e^f, \quad d = U^{-3} V^{-1} e^{-f}.$$

The products UV^3 and $U^{-3}V^{-1}$ may be reduced to unity by a transformation of the group (T). We shall find such a transformation. The effect of a transformation of the group (T) on the coefficients of the system (D) is the following

[1, p. 20], [1, p. 23]:

$$\begin{aligned} \lambda \beta_v \bar{m} &= \mu m, & \mu \alpha_u \bar{n} &= \lambda n, \\ \lambda \alpha_u^2 \bar{a} &= a\lambda + c\lambda_u - \lambda_{uu}, & \mu \beta_v^2 \bar{b}' &= b'\mu + d'\mu_v - \mu_{vv}, \\ \lambda \alpha_u^2 \bar{b} &= \mu[b + d(\log \mu)_v], & \mu \beta_v^2 \bar{a}' &= \lambda[a' + c'(\log \lambda)_u], \\ \alpha_u \bar{c} &= c - (\log \lambda^2 \alpha_u)_u, & \beta_v \bar{d}' &= d' - (\log \mu^2 \beta_v)_v, \\ \lambda \alpha_u^2 \bar{d} &= \mu \beta_v d, & \mu \beta_v^2 \bar{c}' &= \lambda \alpha_u c'. \end{aligned}$$

From the next to last pair of equations we find

$$\bar{f} = f - \log k'^4 \lambda^2 \mu^2 \alpha_u \beta_v,$$

where k' is an arbitrary constant. The relations

$$\begin{aligned} \bar{m} &= (\mu/\lambda \beta_v) UV^3 e^f = e^{\bar{f}} = e^f (k'^4 \lambda^2 \mu^2 \alpha_u \beta_v)^{-1}, \\ \bar{d} &= (\mu \beta_v / \lambda \alpha_u^2) (U^3 V e^f)^{-1} = e^{-\bar{f}} = e^{-f} k'^4 \lambda^2 \mu^2 \alpha_u \beta_v, \end{aligned}$$

show that the conditions

$$(10) \quad UV^3 = (k'^4 \lambda^3 \alpha_u)^{-1}, \quad U^3 V = (k'^4 \lambda^3 \mu \alpha_u^3)^{-1}$$

must be satisfied. If the conditions (10) are solved for U and V , there results the equations

$$(11) \quad k' \mu V = 1, \quad k' \lambda \alpha_u U = 1.$$

The conditions (11) and hence also the conditions (10), are satisfied by a transformation of the form

$$(12) \quad k' \lambda U = 1, \quad k' \mu V = 1, \quad \alpha_u = \beta_v = 1.$$

When this transformation has been made, we have

$$\bar{m} = e^{\bar{f}}, \quad \bar{d} = e^{-\bar{f}},$$

which we may write as

$$(13) \quad m = e^f, \quad d = e^{-f},$$

Therefore, we may replace the conditions (7) by (13) so that the conditions (13) are necessary and sufficient conditions that the surface S_y be a quadric. The equations (11) show that α_u and β_v are arbitrary. In fact, a transformation of the group (T) exists such that the equations

$$m = e^f, \quad d = e^{-f}, \quad n = e^f, \quad c' = e^{-f},$$

are obtained, if both of the surfaces S_y and S_z are quadrics.

The curves of the parametric net on the surface S_y are, in general, space curves. The condition [1, p. 53]

$$d_1 = 0,$$

is necessary and sufficient that the u -curves on the surface be plane curves. Moreover, the condition

$$c' = 0$$

is the condition which must be satisfied in case the v -curves

are plane curves. The u -curves have been shown to be straight lines if the surface S_y is a developable. The v -curves cannot be straight lines. The coordinates y cannot satisfy a second order differential equation with respect to v as seen by differentiating the first equation of the system (D) once with respect to v and using the facts that z and z_v are distinct and that the inequality

$$m \neq 0$$

must hold.

For the surface S_z we may make statements corresponding to those made for the surface S_y . The equations and conditions written for the surface S_y hold also for the surface S_z by making the substitutions

$$\begin{pmatrix} u, m, a, b, c, d, y \\ v, n, b', a', d', c', z \end{pmatrix}.$$

The differential equations of the Laplace transformed congruences [1, § 10] of the system (D) may be written in the form of system (D). For these equations it is customary to use the same coefficients as are used in the system (D) with the subscript k for the k th Laplace transformed congruence and the subscript $-k$ for the minus k th Laplace transformed congruence where k is a positive integer.

The equations and conditions obtained for the surface $S_y(S_z)$ hold for any plus (minus) Laplace transform of the surface $S_y(S_z)$ by using the proper subscript. The coefficients

with subscripts may be expressed in terms of the coefficients of the system (D). To illustrate, we have for the surface S_p the fact that if

$$d_{,1} = 0,$$

the surface S_p is a developable. From the integrability condition of the system (D_1) which corresponds to the third integrability condition of the system (D), we have, in this case,

$$b_{,1} = m[I_u - I\{\log(d/m)\}_u + b_n + d n_v] = 0.$$

Also, we have for the surface S_p the fact that if

$$c'_{,1} = a'_v - a'(\log c')_v + c'mn = 0,$$

the surface S_p is a developable. From the third integrability condition of the system (D_{-1}) , we have, in this case,

$$a'_{,1} = n[J_v - J\{\log(c'/n)\}_v + a'm + c'm_u] = 0.$$

Any Laplace transform of the surface S_y may be degenerate. As previously stated, the surface S_z cannot degenerate into a u-curve. From the second equation of system (D) we see that the condition

$$n = 0$$

is a necessary condition that the surface S_z degenerate into a y-curve. This condition is also sufficient.

The curve C_z degenerates into a straight line if, and only if, the coordinates of the point P_z satisfy an equation of the form

$$z_{vv} + A_{-,1} z_v + B_{-,1} z = 0.$$

The fourth equation of system (D) gives

$$A_{-1} = -d', \quad B_{-1} = -b', \quad a' = c' = 0.$$

Therefore, the surface S_z degenerates into a straight line if, and only if,

$$n = c' = 0.$$

In a similar way it could be shown that the surface S_p degenerates into a u-curve in case

$$(14) \quad m_1 = 0;$$

and into a straight line in case

$$m_1 = d_1 = 0.$$

Corresponding conditions could also be given for the k th and minus k th Laplace transforms of S_y .

With the exception of the surface S_z any Laplace transform of S_y may degenerate into either a u-curve or a v-curve. We shall now show that the surface S_p may degenerate into a v-curve. The surface S_p degenerates into a curve in case the condition

$$(15) \quad A_1 \rho + B_1 \rho_u + C_1 \rho_v = 0,$$

is satisfied. The equation (15) may be written in the form

$$\alpha_1 y + \beta_1 z + \gamma_1 y_u + \delta_1 z_v = 0,$$

where

$$\alpha_1 = -A_1 (\log m)_u + B_1 [a - (\log m)_{uu}] + C_1 k = 0, \quad \beta_1 = B_1 b = 0,$$

$$\gamma_1 = A_1 + B_1 [c - (\log m)_u] = 0, \quad \delta_1 = B_1 d = 0.$$

Therefore, either

$$A_1 = B_1 = 0, \quad C_1 \neq 0, \quad k = 0,$$

or

$$(16) \quad A, B, \neq 0, \quad d = 0, \quad C, = 0, \quad k \neq 0.$$

The equivalent of the first conditions has been given above in equation (14). If the conditions (16) are satisfied, the equation (15) becomes

$$\rho_u - [c - (\log m)_u] \rho = 0.$$

Hence, the curve C_ρ is a v -curve and the surface S_y is a developable with the u -curves as generators. The curve C_ρ is the edge of regression of the developable surface S_y . Since

$$mk \neq 0$$

and the points P_z and P_{z_v} are distinct, the coordinates do not satisfy a third order differential equation with respect to v and, therefore, the curve C_ρ is a space curve and the surface S_y cannot be a plane. We may, therefore, make the statements that the surface S_ρ degenerates into a v -curve in case

$$k \neq 0, \quad d = 0$$

and that if the surface S_ρ degenerates into a v -curve, the surface S_y is a developable with the u -curves as generators and with the curve C_ρ as its edge of regression. Since the surface $S_y(S_z)$ cannot degenerate into a $v(u)$ -curve, the surface $S_z(S_y)$ cannot be a developable surface with its $u(v)$ -curves as generators.

With the exception of the surfaces S_z , any of the Laplace transforms of S_y of system (D) may reduce to a fixed point. Then the adjacent Laplace transform of the surface

reducing to a point is a cone with the fixed point as vertex. If the k th Laplace transform of S_y reduces to a fixed point, then the k th Laplace transformed congruence consists of straight lines of the cone y_{k-1} and is not a proper congruence.

The point P_p is a fixed point if, and only if, the coordinates of the point satisfy the equations

$$\rho_u + p'\rho = 0, \quad \rho_v + q'\rho = 0, \quad p'_v = q'_u,$$

or

$$\begin{aligned} [a - (\log m)_{uu} - p'(\log m)_u]y + bz + [c - (\log m)_u + p']y_u + dz_v &= 0, \\ [k - q'(\log m)_u]y + q'y_u &= 0, \quad p'_v = q'_u, \end{aligned}$$

where

$$p' = -[c - (\log m)_u], \quad q' = 0.$$

From these equations, we obtain

$$k = d = b = I = 0.$$

Hence, the point P_p is a fixed point if, and only if,

$$k = d = 0.$$

Similarly, the point P_q is fixed in case

$$n_{-1} = c' = 0.$$

CHAPTER II

SYSTEMS OF EQUATIONS OF THE RAY CONGRUENCE

3. The system (Δ). We shall derive a system of differential equations of the ray congruence determined by the parametric conjugate net on the integral surface S_y . We call this system, the system (Δ).

We recall that if n is different from zero, the focal points ϕ_1, ϕ_2 of the ray congruence are given by

$$\phi_1 = \rho + \lambda_1 z, \quad \phi_2 = \rho + \lambda_2 z,$$

where

$$2n\lambda_1 = -I + (I^2 + 4dkn)^{1/2}, \quad 2n\lambda_2 = -I - (I^2 + 4dkn)^{1/2}.$$

Owing to the relations between the surface S_y and its first and minus first Laplace transforms, the same geometrical results are obtained by assuming k to vanish instead of n . If but one of these focal sheets S_z, S_ρ is degenerate, we shall always assume that the degenerate focal sheet is S_ρ . In this chapter, we impose the conditions

$$(1) \quad I^2 + 4dkn \neq 0, \quad n \neq 0.$$

Then the focal sheets ϕ_1, ϕ_2 are distinct. The conditions (1)

also exclude the possibility of either focal sheet reducing to a point, as is shown later.

Since the two focal sheets of the ray congruence have the line ϕ, ϕ_2 for common tangent, we may write

$$(2) \quad \begin{aligned} \phi_{1v} &= A_2 \phi_1 + B_2 \phi_2 + C_2 \phi_{1u}, \\ \phi_{2u} &= A'_2 \phi_1 + B'_2 \phi_2 + D'_2 \phi_{2v}, \end{aligned}$$

where the coefficients $A_2, B_2, \dots, D'_2$ are given by

$$dA_2 = -\lambda_1 [c - (\log m)_u] - X_1/t, \quad dB_2 = X_1/t, \quad dC_2 = \lambda_1,$$

$$\lambda_2 A'_2 = X_2/t, \quad \lambda_2 B'_2 = \lambda_2 [c - (\log m)_u] - X_2/t, \quad \lambda_2 D'_2 = d,$$

and X_1, X_2, t are defined by

$$X_1 = \lambda_1^2 [c - (\log m)_u] - \lambda_1 (b + \lambda_{1u}) + d \lambda_{1v},$$

$$X_2 = \lambda_2^2 [c - (\log m)_u] - \lambda_2 (b + \lambda_{2u}) + d \lambda_{2v},$$

$$t = \lambda_2 - \lambda_1.$$

The coefficients $A_2, B_2, \dots, D'_2$ may be determined by writing the equations (2) each in the form

$$\alpha_2 y + \beta_2 z + \gamma_2 y_u + \delta_2 z_v = 0$$

and by making use of the conditions

$$\alpha_2 = \beta_2 = \gamma_2 = \delta_2 = 0.$$

The equations (2) are the first two equations of the system (Δ). The remaining two equations of this system are

$$(3) \quad \begin{aligned} \phi_{1uu} &= A_3 \phi_1 + B_3 \phi_2 + C_3 \phi_{1u} + D_3 \phi_{2v}, \\ \phi_{2vv} &= A'_3 \phi_1 + B'_3 \phi_2 + C'_3 \phi_{1u} + D'_3 \phi_{2v}, \end{aligned}$$

where the coefficients $A_3, B_3, \dots, D'_3$ are obtained as above.

In order to put some of the subsequent results in a briefer form, we list here the definitions of some notations:

$$(4) \quad \lambda_i \alpha_i = d\alpha_i - nX_i,$$

- (5) $\lambda_i \beta_i' = d\beta_i' + (\log \lambda_i d)_u x_i - x_i u,$
 (6) $\alpha_i' = k_u + (\lambda_i n)_v - k[\log(\lambda_i/m)]_u,$
 (7) $\beta_i' = \lambda_i (\log \lambda_i m)_{uv},$
 (8) $d^2 \alpha_i'' = d\lambda_i \alpha_i' - (I + \lambda_i n) X_i,$
 (9) $d\beta_i'' = \lambda_i \beta_i' - 2(\log \lambda_i)_v X_i + X_i v,$
 (10) $p_i = -c + [\log(m/d)]_u,$
 (11) $q_i = -[I + \lambda_i n + c_u - (\log m)_{uu} - (\log d)_u \{c - (\log m)_u\}],$
 (12) $r_i = 2(\log \lambda_i)_v + d^* \quad (i = 1, 2).$

The coefficients of the equations (3) are now the following:

$$\begin{aligned} tA_3 &= -\beta_i - tq_i + \alpha_i (X_2/\lambda_2 + \lambda_{2u} - \lambda_{1u})/tn + p_i X_1/\lambda_1, \\ tB_3 &= \beta_i - \alpha_i [X_1/\lambda_1 + d\{\log(\lambda_2/\lambda_1)\}_v]/tn - p_i X_1/\lambda_1, \\ C_3 &= -p_i - \alpha_i/tn, \quad D_3 = d\alpha_i/\lambda_2 tn, \\ tA_3^* &= -\beta_2^* + \alpha_2^* (X_2/\lambda_2 + \lambda_{2u} - \lambda_{1u})/tn, \\ tB_3^* &= \beta_2^* + \lambda_2 tc^* - \alpha_2^* [X_1/\lambda_1 + d\{\log(\lambda_2/\lambda_1)\}_v]/tn, \\ C_3^* &= -\alpha_2^*/tn, \quad D_3^* = r_2 + d\alpha_2^*/\lambda_2 tn. \end{aligned}$$

4. The system (D_R). A system of differential equations of the form (D) for the ray congruence may be obtained from the system (Δ). This new system, we shall write as follows:

$$\begin{aligned} \bar{\theta}_{1\bar{v}} &= M\bar{\theta}_1, \quad \bar{\theta}_{2u} = N\bar{\theta}_1, \\ (D_R) \quad \bar{\theta}_{1u\bar{u}} &= A\bar{\theta}_1 + B\bar{\theta}_2 + C\bar{\theta}_{1\bar{u}} + D\bar{\theta}_{2\bar{v}}, \\ \bar{\theta}_{2\bar{v}\bar{v}} &= A'\bar{\theta}_1 + B'\bar{\theta}_2 + C'\bar{\theta}_{1\bar{u}} + D'\bar{\theta}_{2\bar{v}}, \end{aligned}$$

where the coefficients $M, N, A, \dots, D'$ are functions of $\bar{u}$ and $\bar{v}$. The integrability conditions of this system are a con-

sequence of the integrability conditions of the system (D).

The developables of the congruence are obtained by equating $\bar{u}$ and $\bar{v}$ to a constant. The differential equation

$$(1) \quad dmn \, du^2 - mI \, du \, dv - mk \, dv^2 = 0$$

has been given as the equation of the ray curves on the surface S_y . This equation also represents the developables of the ray congruence or also the curves of intersection of the developables of the ray congruence with its focal sheets ϕ_1 and ϕ_2 . Since m , n , and d are different from zero, the equation (1) may be replaced by

$$(d \, du + \lambda_1 \, dv)(d \, du + \lambda_2 \, dv) = 0.$$

Let the coordinates $\bar{u}$ and $\bar{v}$ be functions of u and v which we may write as follows

$$(2) \quad \bar{u} = \zeta(u, v), \quad \bar{v} = \eta(u, v).$$

If the functions ζ and η satisfy the conditions

$$\lambda_1 \zeta_u = d\zeta_v, \quad \lambda_2 \eta_u = d\eta_v \quad (\zeta_u \eta_v - \zeta_v \eta_u) \neq 0,$$

then

$$d^2 \bar{u} \, d\bar{v} = \zeta_u \eta_u (d \, du + \lambda_1 \, dv)(d \, du + \lambda_2 \, dv) = 0,$$

is the differential equation of the parametric net on the surface ϕ_1 and also on the surface ϕ_2 .

If to the first two equations of system (Δ), we apply the transformation (2), we obtain

$$(3) \quad \begin{aligned} t\eta_u \phi_{,\bar{v}} &= A_2 d\phi_1 + B_2 d\phi_2, \\ t\zeta_u \phi_{,\bar{u}} &= A_1 \lambda_2 \phi_1 + B_1 \lambda_2 \phi_2. \end{aligned}$$

The equation (3) may be simplified into

$$(4) \quad t\gamma\gamma_u\bar{\rho}_{i\bar{v}} = B_2d\delta\bar{\rho}_2, \quad t\delta\zeta_u\bar{\rho}_{2\bar{u}} = A_2'\lambda_2\gamma\bar{\rho}_1,$$

by applying the transformation

$$(5) \quad \rho_i = \gamma(\bar{u}, \bar{v})\bar{\rho}_i, \quad \rho_2 = \delta(\bar{u}, \bar{v})\bar{\rho}_2,$$

where

$$\log \gamma = \int (A_2 d/t \gamma_u) d\bar{v}, \quad \log \delta = \int (B_2' \lambda_2 / t \zeta_u) d\bar{u}.$$

The equations (4) are the first two equations of the system (D_R) where

$$t_M \gamma_u \gamma = dB_2 \delta, \quad t_N \zeta_u \delta = \lambda_2 A_2' \gamma.$$

We shall now obtain the coefficients of the remaining two equations of the system (D_R) . By using the transformation (2), the partial derivatives of ρ_i , $(i = 1, 2)$ are found to be transformed according to the equations

$$\begin{aligned} \rho_{iu} &= \rho_i \bar{u} \zeta_u + \rho_i \bar{v} \gamma_u, & \rho_{iv} &= \rho_i \bar{u} \zeta_v + \rho_i \bar{v} \gamma_v, \\ \rho_{i uu} &= \rho_i \bar{u} \bar{u} \zeta_u^2 + 2\rho_i \bar{u} \bar{v} \zeta_u \gamma_u + \rho_i \bar{v} \bar{v} \gamma_u^2 + \rho_i \bar{u} \zeta_{uu} + \rho_i \bar{v} \gamma_{uu}, \\ \rho_{i uv} &= \rho_i \bar{u} \bar{u} \zeta_u \zeta_v + \rho_i \bar{u} \bar{v} (\zeta_u \gamma_v + \zeta_v \gamma_u) + \rho_i \bar{v} \bar{v} \gamma_u \gamma_v + \rho_i \bar{u} \zeta_{uv} + \rho_i \bar{v} \gamma_{uv}, \\ \rho_{i vv} &= \rho_i \bar{u} \bar{u} \zeta_v^2 + 2\rho_i \bar{u} \bar{v} \zeta_v \gamma_v + \rho_i \bar{v} \bar{v} \gamma_v^2 + \rho_i \bar{u} \zeta_{vv} + \rho_i \bar{v} \gamma_{vv}. \end{aligned}$$

The inverse of the above equations is found, by direct calculation, to be

$$\begin{aligned} \Delta \rho_i \bar{u} &= \gamma_v \rho_{iu} - \gamma_u \rho_{iv} = \Delta (\lambda_2 \rho_{iu} - d \rho_{iv}) / \zeta_u (\lambda_2 - \lambda_1), \\ \Delta \rho_i \bar{v} &= -\zeta_v \rho_{iu} + \zeta_u \rho_{iv} = -\Delta (\lambda_1 \rho_{iu} - d \rho_{iv}) / \gamma_u (\lambda_2 - \lambda_1), \\ (6) \quad \lambda_2^2 \Delta^2 \rho_{i \bar{u} \bar{u}} &= \gamma_v^2 [\lambda_2^2 \rho_{i uu} - 2d \lambda_2 \rho_{i uv} + d^2 \rho_{i vv} - \zeta_u \lambda_2^2 \rho_{i \bar{u}} - \gamma_u \lambda_2^2 \rho_{i \bar{v}}], \\ \lambda_1^2 \Delta^2 \rho_{i \bar{v} \bar{v}} &= \zeta_v^2 [\lambda_1^2 \rho_{i uu} - 2d \lambda_1 \rho_{i uv} + d^2 \rho_{i vv} - \zeta_u \lambda_1^2 \rho_{i \bar{u}} - \gamma_u \lambda_1^2 \rho_{i \bar{v}}], \end{aligned}$$

where

$$\begin{aligned} \Delta &= \zeta_u \gamma_v - \zeta_v \gamma_u, \\ \lambda_i^2 \pi_i &= \lambda_i^2 \pi_{uu} - 2d \lambda_i \pi_{uv} + d^2 \pi_{vv}, \quad (\pi = \zeta, \gamma; i = 1, 2). \end{aligned}$$

We shall omit the expressions for $\phi_{i\bar{u}\bar{v}}$ since they are not needed.

If substitutions are made from the equations (6) into the equations

$$(7) \quad \begin{aligned} \phi_{i\bar{u}\bar{u}} &= A_4 \phi_i + B_4 \phi_2 + C_4 \phi_{i\bar{u}} + D_4 \phi_{2\bar{v}}, \\ \phi_{2\bar{v}\bar{v}} &= A'_4 \phi_1 + B'_4 \phi_2 + C'_4 \phi_{i\bar{u}} + D'_4 \phi_{2\bar{v}}, \end{aligned}$$

where the coefficients $A_4, \dots, D'_4$ are functions of $\bar{u}$ and $\bar{v}$, we find the relations

$$\begin{aligned} A_4 &= -B_4 - (\eta_v^2/\Delta^2) \{ q_1 + [c - (\log m)_u] p_1 + 2dk/\lambda_2 - d^2\lambda_1 c'/\lambda_2^2 \\ &\quad + \lambda_1 \lambda_2 [c - (\log m)_u] [\xi_2/\xi_v - \eta_2/\eta_v + (C_4/\xi_v - D_4/\eta_v) \Delta^2/\eta_v^2]/td \}, \\ B_4 &= \lambda_2 [X_1 C_4/\xi_v - (X_{12} + td\lambda_{1v}/\lambda_1) D_4/\eta_v / td] + [\eta_v^2/t\Delta^2] [\beta_2, \\ &\quad + \lambda_2 X_1 (\xi_2/\xi_v - \eta_2/\eta_v)/td], \\ C_4 &= (\xi_v \eta_v^2/\Delta^2) [\lambda_2 \alpha_{21}/tk - dp_1/\lambda_1 - 2d\lambda_{1u}/\lambda_1 \lambda_2 + d^2 r_1/\lambda_2^2 - \xi_2/\xi_v], \\ D_4 &= -\eta_v^3 \lambda_1 \alpha_{21}/tk \Delta^2, \end{aligned}$$

where

$$\begin{aligned} \lambda_i^2 \alpha_{ij} &= \lambda_i^2 \alpha_j - 2d\lambda_i \alpha'_j + d^2 \alpha''_j, \\ \lambda_i^2 \beta_{ij} &= \lambda_i^2 \beta_j - 2d\lambda_i \beta'_j + d^2 \beta''_j, \\ X_{ij} &= \lambda_i^2 [c - (\log m)_u] - \lambda_i (b + \lambda_{ju}) + d\lambda_{jv} \quad (i, j = 1, 2; i \neq j). \end{aligned}$$

The last four coefficients may be obtained from the first four coefficients by making the substitutions

$$\left(\begin{array}{l} \xi_v, \lambda_1, \xi_1, \xi_2, r_1, q_1, \alpha_{12}, \beta_{12}, X_{12}, A_4, B_4, C_4, D_4 \\ \eta_v, \lambda_2, \eta_2, \eta_1, r_2, q_2, -\alpha_{21}, -\beta_{21}, X_{21}, B'_4, A'_4, D'_4, C'_4 \end{array} \right)$$

except in B'_4 the terms ξ_2, η_2, D_4, C_4 change into $-\xi_1, -\eta_1, -C'_4, -D'_4$, respectively.

If the transformation (5) is applied to equations (7), we obtain the last two equations of system (D_R) where

$$\gamma A = \gamma A_4 + \gamma_{\bar{u}} C_4 - \gamma_{\bar{u}\bar{u}}, \quad \gamma B = \delta B_4 + \gamma (\delta_{\bar{v}} / \delta) D_4,$$

$$\gamma C = \gamma C_4 - 2\gamma_{\bar{u}}, \quad \gamma D = \delta D_4, \quad \delta A' = \gamma A'_4 + \gamma_{\bar{u}} C'_4,$$

$$\delta B' = \delta B'_4 + \delta_{\bar{v}} D'_4 - \delta_{\bar{v}\bar{v}}, \quad \delta C' = \gamma C'_4, \quad \delta D' = \delta D'_4 - 2\delta_{\bar{v}}.$$

Facts concerning the ray congruence and its focal sheets may be obtained from the invariants of system (D_R) . Some of these invariants are M, N, D, C' . We know the geometric significance of the vanishing of the coefficients m, n, d, c' of system (D) . Correspondingly for system (D_R) , if M is zero, the surface $\mathcal{P}_1$ degenerates into a curve; if N is zero, the surface $\mathcal{P}_2$ degenerates into a curve. If D is zero, the surface $\mathcal{P}_1$ is a developable. If C' is zero, the surface $\mathcal{P}_2$ is a developable. If both M and D are zero, the curve $\mathcal{P}_1$ is a straight line. If N and C' are zero, the curve $\mathcal{P}_2$ is a straight line.

CHAPTER III

SPECIAL TYPES OF RAY CONGRUENCES

5. Introductory remarks. We shall obtain, in terms of the coefficients of system (D), necessary and sufficient conditions that the focal sheet ϕ , degenerate into a point, a curve, and a straight line. Corresponding conditions are obtainable for the focal sheet ϕ_2 by changing subscript 1 to subscript 2 in the equations pertaining to the focal sheet ϕ . These conditions, we shall use in discussing the nature of conjugate systems which determine ray congruences of special types and in discussing surfaces sustaining such conjugate systems.

6. Bundle of Lines. Let us suppose that one focal sheet of the ray congruence reduces to a fixed point. For n different from zero, the focal sheet ϕ , degenerates into a point if, and only if, the coordinates ϕ , satisfy the equations

$$(1) \quad \phi_{,u} + p \phi = 0, \quad \phi_{,v} + q \phi = 0 \quad p_v = q_u,$$

where p, q are functions of u and v . These equations may be

written as

$$\begin{aligned} [k - q(\log m)_u]y + (\lambda_v + q\lambda_1)z + qy_u + \lambda_1 z_v &= 0, \\ [a - (\log m)_{uu} + \lambda_1 n - p(\log m)_u]y + (b + \lambda_{1u} + p\lambda_1)z \\ + [c - (\log m)_u + p]y_u + dz_v &= 0, \end{aligned}$$

so that

$$(2) \quad q = \lambda, \quad k = d = b = I = 0, \quad p_v = -[c - (\log m)_u]_v = 0.$$

Equations (2) and the corresponding equations for $\mathcal{S}_2$ show that if one focal sheet of the ray congruence degenerates into a point, the other focal sheet and the first Laplace transform of S_y also degenerate into points; these three points are coincident and the surface S_y is a cone with the coincident points as vertex. Owing to the symmetric rôle played by the first and the minus first Laplace transforms of S_y , the statement could be modified by replacing the words the first by the words either the first or the minus first. In either case, the ray congruence is a bundle of lines with its center at the focal point.

We shall assume, throughout this section, that the Laplace transform of S_y which reduces to a point is the Laplace transform S_ρ . Then the u-curves on the cone S_y are the generators and the v-curves may be any one-parameter family of curves distinct from the generators. No condition is imposed upon the non-degenerate minus first Laplace transform S_z except that the u-curves on this surface are plane curves. In fact, these curves lie in the tangent planes of the cone S_y . We

shall impose further conditions upon the surface S_z : first, that its v -curves are plane curves; second, that its v -curves are straight lines.

If the v -curves on S_z are plane curves, the surface S_z is non-developable and the surface S_ρ is developable. Therefore,

$$c' \neq 0, \quad c'_{,1} = c' [c' d - (\log c')_{uv}] = 0,$$

from which we obtain the fact that c' is a product U, V , where U is a function of u alone and V is a function of v alone.

The product U, V may be reduced to unity by applying, to system (D), the transformation

$$\lambda = U^{-1}, \quad \mu = V, \quad \alpha_u = \beta_v = 1,$$

which is a transformation of the group (T). When this transformation has been made, the seventh integrability condition reduces to

$$f_{uu} = nJ + a.$$

The invariant J is, in general, different from zero. The axis curves on S_z which is non-developable are given by the equation

$$c' dn du^2 + nJ du dv - c'_{,1} dv^2 = 0.$$

Since

$$d = c'_{,1} = 0, \quad J \neq 0,$$

the axis curves on S_z are coincident with the parametric curves on the surface. If also

$$J = 0,$$

then the axis curves are indeterminate. It has been shown [3,

p. 61] that if the axis curves on a non-developable surface are indeterminate, the first and the minus first Laplace transforms of that surface are cones with a common vertex. Therefore, the surface S_ρ is, in this case, a cone and the minus third Laplace transform of S_γ reduces to a point which is the vertex of the cone S_ρ and coincides with the point P_ρ . It has been shown [3, p. 61] also that if the axis curves on a non-developable surface are indeterminate, the axis congruence is a bundle of lines through the common vertex of the two cones. It may be noted here that if the first and the minus first Laplace transforms of a given surface are cones with a common vertex, the axis congruence of the net on the given surface and the ray congruences of the nets on the two cones are the same congruence and the lines of this congruence form a bundle of lines through the common vertex of the two cones.

If the v -curves on the surface S_z are straight lines, the surface S_z is a developable and the v -curves on the surface S_γ are plane curves. The surface S_ρ becomes degenerate and is the edge of regression of the developable surface S_z . The edge of regression is, in general, a space curve as stated in section 2. The curve C_ρ may, however, degenerate into a fixed point. Then the developable surface S_z is a cone with the fixed point P_ρ as vertex. The fixed points P_ρ and P_ρ are not coincident since linear combinations of their co-

ordinates

$$\rho = y_u - (\log m)_u y, \quad \sigma = z_v - (\log n)_v z,$$

do not exist.

Let us now assume that n is zero. Then the equations (1) and (2) are not valid. Since n is zero, the surface S_z is degenerate. If, however, the surface S_ρ reduces to a fixed point and the surface S_z degenerates into a curve, the rays form a pencil of lines through the fixed point. Then

$$n = k = d = 0.$$

Hence, the surface S_y is a cone with the fixed focal point as vertex. The u -curves are the generators and the v -curves are the curves whose tangent developables have the curve C_z in common. The v -curves are plane curves if the curve C_z is a straight line.

It may happen, also, that the surfaces S_ρ and S_z reduce to distinct fixed points, in which case, there is but a single ray. Then system (D) does not hold. The coordinates of the point P_y satisfy the Laplace equation

$$(3) \quad y_{uv} = ay_u + by_v + cy,$$

where the coefficients a , b , c , are analytic functions of u and v . Equation (3) may be written in the two forms

$$(4) \quad \rho_v = a\rho + Ky, \quad z_u = bz + Hy,$$

where

$$(5) \quad \begin{aligned} \rho &= y_u - by, & z &= y_v - ay, \\ H &= ab + c - a_u, & K &= ab + c - b_v. \end{aligned}$$

From the equations (4) and the fact that the points P_ρ and P_z

are fixed, we have

$$(6) \quad H = K = 0, \\ \rho_u = d\rho, \quad z_v = ez, \quad d_v = a_u, \quad e_u = b_v.$$

Since H and K vanish, the transformation

$$y = \lambda \bar{y}$$

reduces equations (3), (5), and (6) to the equations

$$\bar{y}_{uv} = 0, \quad \bar{\rho} = \bar{y}_u, \quad \bar{z} = \bar{y}_v, \\ \bar{y}_{uu} = d\bar{y}_u, \quad \bar{y}_{vv} = e\bar{y}_v, \quad \bar{d}_v = \bar{e}_u = 0.$$

Hence, if the first and the minus first Laplace transforms of S_y reduce to distinct fixed points, the surface S_y is a plane and the parametric net consists of straight lines through the two fixed points.

7. Ruled plane. We shall now suppose that the lines of the ray congruence lie in a fixed plane. If the lines of a ruled plane are the rays determined by a conjugate net, the ruled plane is called [4, p. 231] the ray plane of the net. The first and the minus first Laplace transforms of S_y cannot be planes since the surface S_y cannot be a plane. Hence, these Laplace transforms are degenerate so that

$$k = n = 0.$$

If also $d = 0$, the point P_ρ is fixed, and this leads to cases discussed in the preceding section. Hence,

$$d \neq 0.$$

The points $P_z, P_{z_v}, P_\rho, P_{\rho_u}$ lie in the ray plane of the net and, therefore,

$$|z \ z_v \ \rho \ \rho_u| = I = 0.$$

The points P_z, P_{z_v}, P_ρ are not collinear and may be considered as determining the plane. This plane is fixed since

$$(z \ z_v \ \rho)_u = [c - (\log m)_u](z \ z_v \ \rho),$$

$$(z \ z_v \ \rho)_v = d^*(z \ z_v \ \rho).$$

The last equation depends upon the conditions

$$k = l^* = 0.$$

From the conditions $k = I = 0$ and the eighth integrability condition of system (D), we obtain

$$l^* = 0.$$

Therefore, the conditions

$$(1) \quad k = n = I = 0$$

are necessary and sufficient conditions that the ray congruence, determined by a conjugate net on a non-developable surface S_y , form a ruled plane. Since the foci of the ray are now any two points on the ray which separate the points P_ρ and P_z harmonically, the focal points of the ray congruence are the points of the ruled plane.

The conditions (1) are also necessary and sufficient conditions that the ray curves on the surface S_y be indeterminate. Conjugate systems with indeterminate ray curves have been studied [5] in the case in which S_y is non-developable. It was found that the parametric curves on S_y are cone curves. Quadric surfaces were found to be a subclass of surfaces on which there exists a conjugate system with indeterminate ray curves. It was found also that all surfaces on

which there exists a conjugate net with indeterminate ray curves can be projected into a surface of translation. Surfaces of translation may be characterized [4, p. 231] as the only surfaces that have the property that on each of them there exists a conjugate net with indeterminate ray curves and with ray plane at infinity.

The curves C_z and C_p may be straight lines. Then al-

so

$$c' = 0, \quad d, = d[c' d - (\log d)_{uv}] = 0,$$

or

$$c' = (\log d)_{uv} = 0.$$

In this case, the parametric curves on the surface S_y are plane curves. The axis curves, which are given by the equation

$$d, du^2 - mI du dv - c' dm dv^2 = 0,$$

are indeterminate. Hence, the axis congruence is a bundle of lines with center on the common vertex of the two degenerate cones, l_z and l_p . From the conditions $n = k = 0$, we have

$$(\log m)_{uv} = 0.$$

Since

$$f_{uv} = mn - c' d = 0, \quad f_{,uv} = f_{uv} - [\log(m/d)]_{uv} = 0,$$

the congruences of systems (D) and (D₁) are W congruences, and the parametric net on the surface S_y is an R net [14, p. 1078]. Since every R net is isothermally conjugate, or since

$$[\log (d/m)]_{uv} = 0,$$

the parametric net is isothermally conjugate [2, p. 322].

Since the surface S_y contains an R net, S_y is an R surface.

8. Ray congruences with degenerate focal sheets. We shall impose the condition that the focal sheets of the ray congruence reduce to curves which are not necessarily straight lines. Special cases in which both focal sheets reduce to straight lines will be considered in Chapter IV.

The focal sheet ϕ_1 degenerates into a curve if, and only if, the coordinates ϕ_1 satisfy the equation

$$(1) \quad A'_1 \phi_1 + B'_1 \phi_{1,u} + C'_1 \phi_{1,v} = 0,$$

where A'_1 , B'_1 , C'_1 are functions of u and v such that B'_1 and C'_1 are not both zero. Equation (1) may be written in the form

$$\alpha y + \beta z + \gamma y_u + \delta z_v = 0.$$

The coefficients $\alpha, \beta, \gamma, \delta$ are functions of A'_1, B'_1, C'_1 and of the coefficients of system (D). From the conditions

$$\alpha = \beta = \gamma = \delta = 0,$$

we find equation (1) to be the first of equations (2) in section 3 with the restriction that X_1 must vanish. Hence,

$$X_1 = 0$$

is a necessary and sufficient condition that the surface ϕ_1 degenerate into a curve. For $n \neq 0$, the corresponding condition for ϕ_2 is obtainable by changing subscript 1 to subscript 2.

The curves ϕ_1 and ϕ_2 may be distinct or coincident. Let us suppose first that these curves are distinct. Then

$$(2) \quad I^2 + 4dkn \neq 0.$$

Therefore, the surface S_y is non-developable if the focal sheets of the ray congruence degenerate into focal curves which are distinct. The ray curves now form a proper net. The developables of the ray congruence are cones. The vertex of each cone is a point on one of the focal curves and the directrix of the cone is the other focal curve.

For $n \neq 0$, we have, from the conditions

$$X_1 = 0, \quad X_2 = 0,$$

the equations

$$(3) \quad n^2(X_1 + X_2) = I^2[c - \{\log(mI/n)\}_{\mathbf{u}}] + 2dkn[c - \{\log(m^2dk/n)\}_{\mathbf{u}}]^{1/2} \\ - dnI[d' + \{\log(dI/n)\}_{\mathbf{v}}] = 0,$$

$$(4) \quad n[(X_1/\lambda_1) + (X_2/\lambda_2)] = -I[c - \{\log(mI/n)\}_{\mathbf{u}}] + 2dn[d + \{\log(d^3k/n)\}_{\mathbf{v}}]^{1/2} = 0.$$

We shall show that the ray curves on S_y are now cone curves. To do this, we shall find the curves on S_y whose tangent developables pass through a focal curve of the ray congruence. Let these curves be called the u' -curves and the v' -curves. The curves conjugate to the u' -curves and the v' -curves are cone curves. Let

$$u' = \zeta'(u, v), \quad v' = \eta'(u, v) \quad \zeta'_u \eta'_v - \zeta'_v \eta'_u = 0.$$

Then, if the conditions

$$y_{u'} + A_1'' y = B_1'' \delta, \quad y_{v'} + A_2'' y = B_2'' \delta$$

are to be satisfied where $A_1'', B_1'', A_2'', B_2''$ are functions of u', v' , we must have

$$(5) \quad m \eta'_u = -\lambda_1 \eta'_v, \quad m \zeta'_u = -\lambda_2 \zeta'_v,$$

respectively. Therefore, the curves given by

$$(6) \quad \lambda_1 du - m dv = 0, \quad \lambda_2 du - m dv = 0,$$

are the curves whose tangent developables pass through the curves β_1 and β_2 , respectively. But the curves conjugate to the curves given by equations (6) are those given by

$$(7) \quad d\,du + \lambda_1\,dv = 0, \quad d\,du + \lambda_2\,dv = 0,$$

respectively. The equations (7) represent the two families of ray curves.

If the ray curves form a conjugate net, then

$$(\log m)_{uv} = 0,$$

and the equations (6) become the equations (7) in reverse order, since then

$$\lambda_1\lambda_2 = -dm.$$

We have shown, for n not zero, that if the ray curves on the surface S_y form a conjugate net, the degenerate focal sheets of the ray congruence are Laplace transforms of S_y referred to its ray curves. This is true also for n equal to zero.

Then

$$k = 0, \quad Id \neq 0,$$

so that

$$\beta_1 = \rho, \quad \beta_2 = z, \quad \rho_v = z_u = 0, \quad (\log m)_{uv} = 0.$$

In this case, the ray curves form a conjugate net coincident with the parametric net. The u -curves and the v -curves on S_y are cone curves.

The surface S_y is, in general, non-ruled. It may be shown that if the ray congruence of a non-developable surface has degenerate focal sheets and the ray curves coincide either with the asymptotic net or with the associate conju-

gate net, the surface S_y is a quadric. If the ray curves and the curves of the asymptotic net are coincident, the conditions

$$(8) \quad I = 0, \quad k = -mn \neq 0, \quad \lambda_1 = -\lambda_2 = (-dm)^{1/2},$$

must be satisfied. From the equations (3), (4), and (8), we obtain

$$(9) \quad B(y) = 0, \quad C''(y) = 0.$$

The equations (9) are necessary and sufficient conditions that the surface S_y be a quadric. If the ray curves coincide with the associate conjugate net on S_y , the conditions

$$(10) \quad I = 0, \quad k = mn \neq 0, \quad \lambda_1 = (-\lambda_2) = (dm)^{1/2},$$

must be satisfied. The equations (3), (4), and (10) lead to equations (9).

Let us assume that the curves ϕ_1 and ϕ_2 are coincident. It has been shown [6, p. 86] that if the focal sheets of a congruence degenerate into two curves which coincide, this focal curve lies on a developable surface whose tangent planes along the points of the curve are the developables of the congruence. The lines of the congruence form flat pencils with centers on the focal curve. For coincident focal curves, we have the conditions

$$(11) \quad I^2 + 4dkn = 0, \quad n \neq 0.$$

We shall consider the cases

$$(12) \quad dkn \neq 0,$$

$$(13) \quad d = I = 0, \quad kn \neq 0,$$

$$(14) \quad k = I = 0, \quad dn \neq 0,$$

for which the conditions (11) are satisfied.

If the conditions (12) hold, then also

$$\rho_1 = \rho_2 \neq \rho \neq z,$$

$$X_1 = X_2 = \lambda_1^2 [c - \{ \log(dm^2 k/n) \}_{u}^{1/2} \pm (-dn/k)^{1/2} (d' + \{ \log(d^3 k/n) \}_{v}^{1/2})] = 0.$$

In this case, the surfaces S_x and S_ρ are restricted to be non-degenerate and the surface S_y is, in general, non-ruled.

If

$$(\log m)_{uv} = 0,$$

then

$$X_1 = X_2 = \lambda_1^2 [B^{(y)} \pm (-d/m)^{1/2} C''^{(y)}] = 0,$$

and the surface S_y is ruled.

If the conditions (13) hold, then

$$\rho_1 = \rho_2 = \rho, \quad \rho_u - [c - (\log m)_u] \rho = 0, \quad X_1 = 0, \quad X_2 = 0,$$

so that the first Laplace transform S_ρ reduces to a curve with which the focal curves coincide. The surface S_y is, in this case, a developable surface with its u-curves as generators and with the curve C_ρ as its edge of regression. The v-curves are, in general, space curves. The curve C_ρ is a space curve since the surface S_y cannot be a plane. The developables of the ray congruence are planes tangent to the developable surface S_y at points of the curve C_ρ . Since $d = I = 0$, the ray curves on the developable S_y are the u-curves which are the generators. Since the u-curves on the surface S_y are straight lines, the u-curves on the surface S_z are plane curves. No other condition is imposed upon the surface S_z ; if this surface is a developable, the v-curves on

the surface S_y are plane curves. Then also

$$c' = 0$$

and the seventh integrability condition of system (D) becomes

$$J = 0.$$

The edge of regression of S_z is the curve C_ρ . The foci of the ray of any point P_z on S_z are given [2, p. 318] by

$$c' n_{-1} y^2 + n J y \sigma - m n \sigma^2.$$

Since c' and J are zero, the focal sheets of the ray congruence determined by the parametric net on S_z are curves coincident with the curve C_ρ .

The line $l_{\rho\sigma}$ is defined to be the joint ray. The totality of lines $l_{\rho\sigma}$ is the joint ray congruence. The developables of this congruence correspond to curves on the surface S_y , called the joint ray curves represented [2, p. 327] by the differential equation

$$(15) \quad m n_{-1} I \, du^2 - [c' \, d m n \{f_u + (\log [c'/m])_u\} \{f_v + (\log [d/n])_v\} \\ - m_{-1} n_{-1} - m n I J] \, du \, dv + m_{-1} n J \, dv^2 = 0.$$

In the above case, the joint ray curves coincide with the parametric curves on the surface S_y .

If the conditions (14) are satisfied, we have

$$\rho_1 = \rho_2 = \rho, \quad \rho_v = 0, \quad X_1 = X_2 = 0,$$

so that again the first Laplace transform degenerates into a curve C_ρ with which the focal curves coincide. But the curve C_ρ is, in this case, a u -curve distinct from the surface S_y . The surface S_y is, in this case, non-developable. Since $k = I = 0$, the ray curves on S_y coincide with the v -curves.

The v-curves are cone curves since the first Laplace transform S_ρ is degenerate. As a point on the surface S_y moves along a v-curve, the corresponding ray generates a developable intersecting the surface S_z in a v-curve. Let us assume first that the v-curves on S_z , which are now plane curves, are not straight lines. Then the conditions

$$(16) \quad c' \neq 0, \quad c'_{-1} = 0$$

hold, and the surface S_ρ is a developable. The curve C_ρ is on the developable surface S_ρ since the equation

$$\sigma_v - [d' - (\log n)_v] \sigma = c' \rho$$

is satisfied, as may be shown, by making use of the equations (16) and of the equations

$$l' = 0, \quad J = 0,$$

which are obtainable from the seventh and the eighth of the integrability conditions (I'). The developables of the ray congruence are the tangent planes of the developable surface S_ρ . The joint rays, in this case, are the generators of S_ρ . If, now, the v-curves on S_z are straight lines, the v-curves on the surface S_y are plane curves, the surface S_z is a developable, and the surface S_ρ degenerates into the edge of regression of the surface S_z .

CHAPTER IV

LINEAR RAY CONGRUENCES

9. Introduction. Before discussing linear ray congruences, we shall recall a few facts concerning linear congruences.

The lines of a linear congruence intersect two straight lines, called directrices. If the directrices are skew lines, the congruence is a proper congruence and consists of the lines joining each point of one directrix to every point of the other directrix. If the directrices coincide, the congruence is a singular congruence and consists of flat pencils with centers on the coincident directrices. The planes of the flat pencils may be regarded as tangent to a quadric surface along one of its generators. If the directrices intersect, they determine a flat pencil of directrices, and the congruence is a singular congruence which degenerates into a bundle of lines with center at the intersection of the directrices and a ruled plane determined by the directrices. A ray congruence, however, cannot be a linear congruence with

a flat pencil of directrices. If the distinct foci of the rays lie in a plane, the congruence is a ruled plane.

10. Proper congruences. We shall impose the condition that the ray congruence be a proper linear congruence. Since the focal lines are distinct, the condition (2) of section 8 must be satisfied so that the surface S_y is non-developable, and the ray curves form a proper net.

We shall consider the cases

- (1) $k = n = 0, \quad Id \neq 0,$
 (2) $I^2 + 4dkn \neq 0, \quad dn \neq 0.$

If the conditions (1) hold, then also the conditions

- (3) $\phi_1 = \rho, \quad \phi_2 = z, \quad (\log m)_{uv} = 0,$
 $c' = 0, \quad d_1 = d[c'd - (\log d)_{uv}] = 0$

must be satisfied. Hence, the surfaces S_ρ, S_z degenerate into straight lines coincident with the focal lines ϕ_1, ϕ_2 , respectively. In this case, the parametric curves on the surface S_y are plane curves and cone curves. Since

- (4) $n = c' = (\log d)_{uv} = (\log m)_{uv} = 0, \quad I \neq 0,$

the axis curves and the ray curves coincide with the parametric net. Since

$$[\log (d/m)]_{uv} = 0,$$

the parametric net is isothermally conjugate. From the equation (4), we see that

$$f_{uv} = f_{,uv} = 0,$$

and therefore, the congruences of systems (D) and (D_1) are W congruences. Then the parametric net is an R net, and the

surface S_y is an R surface. Since the conditions

$$c_v - (\log m)_{uv} = 0, \quad d'_u + (\log d)_{uv} = 0$$

are satisfied, it follows, as in section 2, that by a transformation (T) given by equations (12) of section 2, we have

$$f = \log m = -\log d,$$

and the surface S_y is a quadric.

Let us now assume the conditions (2) to be satisfied and obtain necessary and sufficient conditions that the surface ϕ_i degenerate into a straight line. The points $\phi_i, \phi_{iu}, \phi_{iv}, \phi_{iuv}, \phi_{iuvv}$ are on the focal line ϕ_i . Therefore, the coordinates of any three of these points are linearly dependent. One of the equations obtainable from this fact is

$$(5) \quad \phi_{iuv} + p_i \phi_{iu} + q_i \phi_i = 0.$$

The equation (5) may be written as

$$\alpha_i y + \beta_i z + \gamma_i y_u + \delta_i z_v = 0,$$

where α_i, β_i are the expressions given by equations (4) and (5) in section 3, and p_i, q_i are the expressions given by equations (10) and (11) in the same section. We have now

$$\alpha_i = \beta_i = 0, \quad \gamma_i = -q_i + q_i = 0, \quad \delta_i = -p_i d + p_i d = 0.$$

The equations (4), (5), (8), (9) in section 3 show that if

$$(6) \quad X_i = \alpha_i = \beta_i = 0,$$

then also

$$(7) \quad \alpha'_i = \beta'_i = \alpha''_i = \beta''_i = 0.$$

The conditions (7) could also have been obtained by setting equal to zero linear combinations of coordinates of points

$\phi_{iuv}, \phi_{iu}, \phi_i$ and of points $\phi_{iuvv}, \phi_{iu}, \phi_i$, respectively,

and proceeding in the above manner. Hence, the surface ϕ_1 degenerates into a straight line if, and only if, the conditions

$$X_1 = \alpha_1 = \beta_1 = 0,$$

are satisfied.

If the conditions (2), (7), and the corresponding conditions for ϕ_2 hold, we may write

$$(8) \quad \alpha'_1 + \alpha'_2 = k[\log(m^2kn/d)]_u - I_v = 0,$$

$$(9) \quad (\lambda_2\beta'_1 + \lambda_1\beta'_2)/\lambda_1\lambda_2 = (\log \lambda_1\lambda_2m^2)_{uv} = [\log(m^2dk/n)]_{uv} = 0.$$

From the last of equations (9), we obtain the fact that if, in this case, the ray curves form a conjugate net, then also the axis curves form a conjugate net since, if

$$(\log m)_{uv} = 0,$$

then also

$$(\log d)_{uv} = 0.$$

It may be shown also that if the surface S_ρ is now non-degenerate, the surface S_z is non-developable. To do this, we note that from the last of equations (8) and the eighth integrability condition (I'), we have

$$(10) \quad d1' + k[c - \{\log(d/n)\}_u] = 0.$$

From the conditions $\alpha''_1 = 0$ and the corresponding condition for ϕ_2 , we have

$$(11) \quad \alpha''_1 + \alpha''_2 = - (I/n)1' - 2k[d' + \{\log(d/n)\}_v] = 0.$$

Then the conditions (10) and (11) show that if c' is zero, the equations

$$(12) \quad c - [\log(d/n)]_u = 0, \quad d' + [\log(d/n)]_v = 0,$$

hold. If the equations (12) are differentiated partially

as to v and to u , respectively, and compared, we obtain

$$f_{uv} = 0.$$

Since now

$$f_{uv} = mn,$$

also n must vanish, which contradicts conditions (2). Hence our assertion is proved. From this it follows that if S_z is developable, then S_p is degenerate.

If could be shown that if the surface S_p is degenerate, the surface S_z is developable, but we shall merely indicate a method of doing this. Let S_z be non-developable, then from equations (8), (10), the seventh condition of (I'), and

$$k = 0, \quad \alpha_1 = 0, \quad \beta'_2 = 0,$$

we could obtain the fact that the surface S_p degenerates into a straight line coincident with the focal line ϕ_1 . The joint ray net would now coincide with the parametric net which is possible if, and only if, n vanishes.

Let us assume that

$$k = c' = 0.$$

The curve C_p is now a straight line coincident with the focal line ϕ_1 . Hence the u -curves of the parametric net are plane curves and the v -curves are cone curves. The v -curves, in this case, coincide with the v -curves of the ray net. From the equations (8), we see that I_v vanishes so that $\beta'_2 = 0$ reduces to

$$n_{-1} = 0.$$

Since

$$c' = n_{\rho} = 0,$$

the surface S_{ρ} degenerates into a point. The surface S_z is, therefore, a cone with the point F_{ρ} as vertex. The focal line ϕ_2 passes through the point F_{ρ} , since

$$\phi_{2v} = -(I/n)\sigma.$$

The planes of the axial pencil with line ϕ_2 as axis intersect the surface S_z in its generators. Since the line l_{ρ} is a straight line, then d_1 is zero, and hence also

$$(\log d)_{uv} = 0$$

so that the axis curves on S_y form a conjugate net.

If the conditions (2), (6) for ϕ_1 and ϕ_2 , and

$$k \neq 0$$

are satisfied, in general, no condition is imposed upon the non-developable surface S_y or upon curves on this surface.

If the ray curves form a conjugate net, the focal lines ϕ_1 , ϕ_2 are first and minus first Laplace transforms of S_y referred to its ray curves. The ray curves are then cone curves and plane curves. The congruences of tangents to the $\bar{u}$ -curves and to the $\bar{v}$ -curves of the ray net are W congruences. Then the ray net is isothermally conjugate and is an R net so that the surface S_y is an R surface. Since now

$$[\log (d/m)]_{uv} = 0,$$

the parametric net on S_y is isothermally conjugate. The parametric net or also the ray net may be regarded as determining a pencil of isothermally conjugate nets which includes its associate conjugate net [7, p. 218].

If instead of imposing the condition that the ray curves form a conjugate net, we impose the condition that the focal points ϕ , and ϕ_1 separate the ray points harmonically, we obtain the following consequences. Since now $I = 0$, the parametric net on the surface S_y is a harmonic conjugate net [7, p. 215]. By differentiating the last of equations (8) with respect to v , (3) in section 8 with respect to v , and (4) in section 8 with respect to u , and comparing with the last of equations (9), we obtain

$$(13) \quad (\log n)_{uv} = (\log d)_{uv} = (\log m)_{uv} = -(\log k)_{uv}^{1/2}, \quad f_{uv} = 0.$$

From equations (13) and from the fifth integrability condition of system (D), we obtain

$$(14) \quad (\log n)_{uv} = (\log c')_{uv}.$$

Equations (13) and (14) show that

$$f_{uv} = 0, \quad f_{1uv} = f_{uv} + [\log(d/m)]_{uv} = 0,$$

$$f_{-1uv} = f_{uv} + [\log(c'/n)]_{uv} = 0,$$

and, therefore, the congruence of systems (D) , (D_1) , (D_{-1}) are W congruences. It has been shown [2, p. 322] that, if the first Laplace transform of a W congruence is a W congruence, the same is true of all its Laplace transforms, and each of these congruences determines by means of its developables an isothermally conjugate system on each of its focal surfaces. Hence, in this case, the parametric net on S_y and its Laplace transforms determine on these surfaces pencils of isothermally conjugate nets.

Since $I = 0$, then $\lambda_1 = -\lambda_2$ so that the conditions

$$\alpha_1 + \alpha_2 = 0, \quad \alpha_1'' + \alpha_2'' = 0$$

where

$$\begin{aligned} \alpha_1 &= (I + \lambda_1 n)_u - (I + \lambda_1 n)[\log(d/m)]_u \\ &\quad - \lambda_1 n [c - (\log m)_u] + \lambda_{1,u} n + \ln = 0, \\ \alpha_1'' &= k_v - kd' - 2k(\log \lambda_1)_v + 1' \lambda_1 = 0, \end{aligned}$$

give

$$(15) \quad l = 0, \quad l' = 0.$$

The axis of the point P_y on the surface S_y intersects the line $l_{x\sigma}$ in the point [2, p. 314]

$$(16) \quad z_y = (1/d) z + \sigma,$$

and the axis of the point P_z on the surface S_z intersects the line $l_{y\rho}$ in the point

$$(17) \quad \tau_y = (1'/c')y + \rho.$$

From equations (15), (16), (17), we obtain the fact that the ray (axis) congruence determined by the parametric conjugate net on the surface $S_y(S_z)$ is the axis (ray) congruence determined by the parametric net on the surface $S_z(S_y)$.

If besides the condition $I = 0$, we impose the further condition

$$(\log m)_{uv} = 0,$$

then from the equations (13), we see that the ray curves and the axis curves on the surfaces S_y and S_z are conjugate nets. It has been shown [3, p. 221] that if the axis curves on two consecutive non-degenerate non-developable surfaces of the Laplace sequence form conjugate nets, then the axis curves on

all of the non-degenerate non-developable Laplace transforms are conjugate nets. The dual statement also holds. Therefore, in this case, the ray curves and the axis curves on the surface S_y and on the Laplace transforms of S_y are conjugate nets.

Since

$$I = 0, (\log m)_{uv} = (\log d)_{uv} = 0,$$

the ray curves, the axis curves, the anti ray curves [7, p. 310], and the anti axis curves [7, p. 310] coincide with the curves of the associate conjugate net. From the sixth integrability condition of (I') , we obtain

$$J = 0.$$

Since

$$J = 0, (\log n)_{uv} = (\log c')_{uv} = 0,$$

the ray curves, the axis curves, the anti ray curves, and the anti axis curves on the surface S_z , coincide with the curves of the associate conjugate net on that surface. In fact, it could be shown that corresponding coincidences occur on each surface of the sequence.

The conditions

$$f_{uv} = (\log m)_{uv} = (\log d)_{uv} = 0,$$

as previously shown, lead to the conditions

$$(18) \quad f = \log m = -\log d$$

so that, the surface S_y is a quadric. Similarly, the conditions

$$f_{uv} = (\log n)_{uv} = (\log c')_{uv} = 0,$$

lead to the conditions

$$(19) \quad f = \log n = -\log c',$$

and the surface S_z is a quadric. It may also be proved that all of the surfaces of the Laplace sequence are quadrics if the conditions (18) and (19) hold. Not merely for the case which we are considering but, in general, if any two consecutive non-degenerate non-developable surfaces of the Laplace sequence are quadrics, then all of the surfaces of the sequence are quadrics. To prove this, we use equations (18) and (19) in the following manner. We have

$$(20) \quad \begin{aligned} f_{,u} &= f_u - (\log m)_u + (\log d)_u = (\log d)_u = -f_u, \\ f_{,v} &= -(b/d) - (\log m)_v = -(\log m)_v = -f_v, \end{aligned}$$

and

$$(21) \quad \begin{aligned} f_{,u} &= -(a'/c') - (\log n)_u = -(\log n)_u = -f_u, \\ f_{,v} &= f_v - (\log n)_v + (\log c')_v = (\log c')_v = -f_v, \end{aligned}$$

so that

$$f_{,u} = -f - k'_1, \quad f_{,v} = -f - k'_2,$$

where k'_1, k'_2 are arbitrary constants. We have also

$$(22) \quad \begin{aligned} m_{,u} &= m[mn - (\log m)_{uv}] = e^f[e^{2f} - f_{uv}] = e^{-f} = 1, e^{f_1}, \\ d_{,u} &= d[c'd - (\log d)_{uv}] = e^{-f}[e^{-2f} + f_{uv}] = e^f = 1', e^{-f_1}, \\ n_{,u} &= n[mn - (\log n)_{uv}] = e^f[e^{2f} - f_{uv}] = e^{-f} = 1, e^{f_1}, \\ c'_{,u} &= c'[c'd - (\log c')_{uv}] = e^{-f}[e^{-2f} + f_{uv}] = e^f = 1', e^{-f_1}, \end{aligned}$$

where $1, 1', 1_1, 1'_1$ are constants. If the first partial derivatives with respect to u and v are found for equations

(22) and if these equations are compared with equations (20) and (21), there results the conditions

$$f_{,u} = (\log m_1)_{,u} = -(\log d_1)_{,u}, f_{,v} = (\log m_1)_{,v} = -(\log d_1)_{,v}, \\ f_{-,u} = (\log n_{-1})_{,u} = -(\log c_{-1}^1)_{,u}, f_{-,v} = (\log n_{-1})_{,v} = -(\log c_{-1}^1)_{,v},$$

Therefore, the equations

$$f_{,u} = (\log m_1^{3/2} d_1^{1/2})_{,u}, \quad f_{,v} = -(\log m_1^{1/2} d_1^{3/2})_{,v},$$

and

$$f_{-,u} = -(\log n_{-1}^{1/2} c_{-1}^{3/2})_{,u}, \quad f_{-,v} = (\log n_{-1}^{3/2} c_{-1}^{1/2})_{,v},$$

are satisfied, so that the surfaces S_ρ and S_σ are quadrics. Since any two adjacent surfaces in the Laplace sequence may be considered as taking the place of the surfaces S_Y and S_Z , we may conclude that all of the surfaces of the sequence are quadrics.

11. Singular congruences. We shall now suppose that the two directrices of the ray congruence coincide. Then
(1) $\phi_1 = \phi_2, \quad \lambda_1 = \lambda_2 = -I/2n = \pm(-dk/n)^{1/2},$
where n is not zero. Since the focal curves are straight lines, also $d \neq 0$. Hence, the surface S_Y is non-developable.

The differential equation of the ray curves now reduces to

$$(2) \quad (2dn \, du - I \, dv)^2 = 0.$$

The developables of the congruence are planes intersecting the surface S_ρ , if non-degenerate, and the surface S_Z in plane curves. We consider the two cases

$$(3) \quad \phi_1 = \phi_2 = \rho, \quad k = I = 0, \quad dn \neq 0,$$

$$(4) \quad \phi_1 = \phi_2 \neq \rho, \quad I^2 + 4dkn = 0, \quad dknI \neq 0.$$

If the conditions (3) are satisfied, the surface S_ρ degenerates into a straight line coincident with the focal line of the ray congruence. The ray congruence and the congruence of system $(D,)$ are W congruences. The line l_ρ is the axis of a one-parameter family of planes which intersect the surface S_y in the u -curves. The conjugate developables of the v -curves on S_y intersect the line l_ρ in a point. Hence, the u -curves on S_y are plane curves and the v -curves on S_y are cone curves. Since $k = I = 0$, $dn \neq 0$, the ray curves on S_y coincide with the v -curves. The developables of the ray congruence are planes which intersect S_z in the v -curves on that surface. If the v -curves on S_z , which are now plane curves, are not straight lines, the surface S_ρ is a developable so that

$$(5) \quad c' \neq 0, \quad c'' = 0.$$

The seventh and the eighth integrability conditions (I') , in this case, reduce to

$$l' = 0, \quad J = 0.$$

Since the surface S_ρ degenerates into a straight line, the conditions

$$(6) \quad b_1 = d_1 = 0$$

must hold. From the conditions (3) and (6), we obtain

$$(7) \quad l = 0.$$

By differentiating equation (7) once with respect to u and making use of the condition $d_1 = 0$, we obtain

$$n_{-1} = 0.$$

Since

$$n_{-1} = c'_{-1} = 0,$$

the surface S_{σ} degenerates into a straight line. The lines l_p and l_{σ} coincide as may be verified from the fact that the equations

$$\begin{aligned} \rho_u &= [c - (\log m)_u] \rho + d \rho, & \rho_v &= 0, \\ \sigma_v &= c' + [d' - (\log n)_v] \sigma, & \sigma_u &= 0 \end{aligned}$$

are satisfied. There is, in this case, a single joint ray.

Since

$$l = l' = 0,$$

the ray congruence determined by the parametric net on $S_y(S_z)$ coincides with the axis congruence determined by the parametric net on $S_z(S_y)$. The ray congruence determined by the parametric net on S_z is the congruence of system (D_1) . As a point on S_y generates a u -curve, the axis through the point generates a cone with its vertex on the line l_p . The axis curves on S_y are, in this case, the u -curves.

Let us assume now that not only the conditions (3) are satisfied but also

$$c' = 0,$$

so that the v -curves on S_z are straight lines. Then, in addition to the statements made in the preceding paragraph regarding the parametric curves on S_y , we may state that in this case, the v -curves on S_y are plane curves. Since now

$$n_{-1} = c' = 0,$$

the surface S_z is a cone and the surface S_ρ degenerates into a fixed point which is the vertex of the cone S_z . From equations (6) and (8), we obtain also

$$(\log d)_{uv} = 0.$$

The conditions

$$c^* = I = (\log d)_{uv} = 0,$$

show that the axis curves on S_y are indeterminate. Hence, the degenerate cone l_ρ has the point P_ρ in common with the cone S_z . The developables of the ray congruence form a pencil of planes intersecting the cone S_z in its generators. This axial pencil also intersects the surface S_y in its u -curves since the axis of the pencil is the line l_ρ .

If the conditions (4) are satisfied, the surfaces S_z and S_ρ are non-degenerate. We shall prove that the parametric curves on the surface S_y form an isothermally conjugate net.

From the conditions

$$X_1 = \alpha_1 = \alpha_1^* = \beta_1^* = 0,$$

we obtain, by making use of equations (1), the conditions

$$(9) \quad \lambda_1 [c - \{\log(m^2 dk/n)\}_{u}^{1/2}] + d[d^* + \{\log(d^3 k/n)\}_{v}^{1/2}] = 0,$$

$$(10) \quad \lambda_1 [c - \{\log(d/n)\}_{u}] + d[d^* + \{\log(d/n)\}_{v}] = 0,$$

$$(11) \quad \lambda_1 [\log(m^2 kn/d)]_{u}^{1/2} + d[\log dkn]_{v}^{1/2} = 0,$$

$$(12) \quad [\log(m^2 dk/n)]_{uv} = 0,$$

respectively. Adding the equations (9) and (11) gives

$$(13) \quad \lambda_1 [c - \{\log(d/n)\}_{u}] + d[d^* + (\log d^2 k)_{v}] = 0.$$

Comparing the equations (10) and (13), gives

$$(14) \quad (\log dkn)_v = 0.$$

Then, from the equations (11), we have also

$$(15) \quad [\log (m^2 kn/d)]_u = 0.$$

Differentiating equation (14) with respect to u , equation (15) with respect to v , and comparing with equation (12), we obtain

$$(\log m)_{uv} = (\log n)_{uv} = (\log d)_{uv} = -(\log k)_{uv}^{1/2}.$$

Since

$$[\log (d/m)]_{uv} = 0,$$

the parametric net on the surface S_y is isothermally conjugate.

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VITA

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